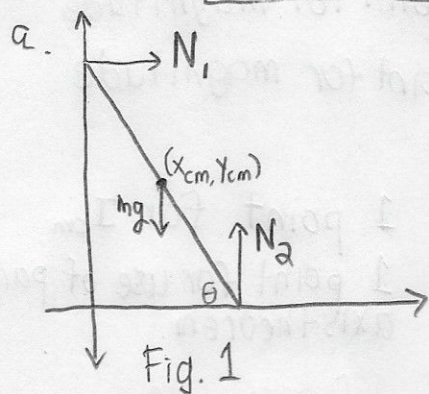PUPC Grading Scheme

Problem 1: 25 points total

a.

Fig. 1
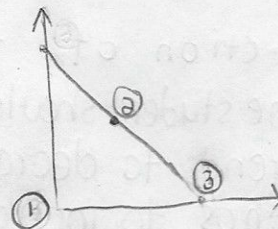
3 points total:
1 point for normal force N_1
1 point for normal force N_2
1 point for weight mg

Deduct $\frac{1}{2}$ from any of these if the force is drawn acting on the wrong point (e.g., if mg is drawn acting on the end of the ladder)

b. Students are free to choose any point. The most sensible are:

1. The corner of the wall and the floor
2. The center of mass of the ladder
3. The end of the ladder resting on the floor

Award $\frac{1}{2}$ point for labeling a point and using it consistently for the rest of the problem.

C. 15 points total (including axioms)

X direction: $m\ddot{x}_{cm} = N_1$ .... (1 point)

Y direction: $m\ddot{y}_{cm} = N_2 - mg$ .... (1.5 points)

Note: if the student uses the opposite sign convention, but fails to note it, deduct $\frac{1}{2}$ point.

Rotational motion: this will depend on the choice made in b)

Method #1: using point 1

All three forces drawn in Fig. 1 contribute to the torque.

Students should introduce a variable such as θ to denote the angle between the ladder and ground (as in Fig. 1).

$$\tau = \tau_{N_1} + \tau_{N_2} + \tau_{mg} = \frac{d\vec{L}}{dt} \qquad \text{(1 point)}$$

$$\tau_{N_1} = 2N_1 \ell \sin\theta \text{ into page} \qquad \text{(1 point for magnitude)}$$
$$\tau_{N_2} = 2N_2 \ell \cos\theta \text{ out of page} \qquad \text{(1 point for magnitude)}$$
$$\tau_{mg} = mg\ell\cos\theta \text{ into page} \qquad \text{(1 point for magnitude)}$$

There are two contributions to the angular momentum:

① $\vec{L}_{rot} = I_{cm}\dot{\theta} \rightarrow \frac{d\vec{L}_{rot}}{dt} = I_{cm}\ddot{\theta}$ into page

where $I_{cm} = \frac{1}{12}m(2\ell)^2 = \frac{1}{3}m\ell^2$ \qquad (1 point)

Deduct $\frac{1}{2}$ point if the student writes $\frac{1}{12}m\ell^2$, forgetting that the length of the rod is 2ℓ.

② $\vec{L}_{cm} = \vec{r}_{cm} \times \vec{p}_{cm}$ \qquad (1 point)

where $\vec{r}_{cm} = (x_{cm}, y_{cm}, 0)$
$$\vec{p}_{cm} = m(\dot{x}_{cm}, \dot{y}_{cm}, 0)$$

So $\vec{L}_{cm} = m(x_{cm}\dot{y}_{cm} - \dot{x}_{cm}y_{cm})$ out of page

$$\frac{d\vec{L}_{cm}}{dt} = m(x_{cm}\ddot{y}_{cm} - \ddot{x}_{cm}y_{cm})$$

At this stage it is a good idea to rewrite x_{cm}, y_{cm} in terms of θ:

$x_{cm} = \ell\cos\theta$ \qquad (1 point)
$\dot{x}_{cm} = -\ell\sin\theta\,\dot{\theta}$
$\ddot{x}_{cm} = -\ell(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)$

$y_{cm} = \ell\sin\theta$
$\dot{y}_{cm} = \ell\cos\theta\dot{\theta}$ \qquad (1 point)
$\ddot{y}_{cm} = \ell(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)$

Putting this all together, (2 points)

$$\tau = -\frac{1}{3}ml^2\ddot{\theta} + ml^2\left(\cos\theta\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right) + \left(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2\right)\sin\theta\right)$$

$$= -\frac{1}{3}ml^2\ddot{\theta} + ml^2\ddot{\theta} = \boxed{\frac{2}{3}ml^2\ddot{\theta} = 2N_2 l\cos\theta - 2N_1 l\sin\theta - mgl\cos\theta}$$

Deduct 1 point if the student uses the incorrect signs.

Method #2: using point 2

The center of mass frame is non-inertial, but any fictitious forces act at the center of mass and therefore do not contribute to the torque; neither does mg.

Therefore, $\quad I_{cm}\ddot{\theta} = \tau_{N_1} + \tau_{N_2}$ (1 point)

$\tau_{N_1} = N_1 l\sin\theta \quad$ into page $\quad$ (15 points for magnitude)

$\tau_{N_2} = N_2 l\cos\theta \quad$ out of page $\quad$ (15 points for magnitude)

$I_{cm} = \frac{1}{12}m(2l)^2 = \frac{1}{3}ml^2 \qquad$ (2 points)

Deduct 1 point if the student writes $I_{cm} = \frac{1}{12}ml^2$

The student should convey that positive $\ddot{\theta}$ corresponds to torque into the page. Therefore:

$$\frac{1}{3}ml^2\ddot{\theta} = N_1 l\sin\theta - N_2 l\cos\theta \qquad \text{(2 points)}$$

Deduct 1 point for an incorrect sign.

Note: if the student uses this method, he or she might defer the calculation of X_{cm}, Y_{cm} to part (d).
In any case, devote 2 points to this.

Method 3: using point 3

This method is enticing because the torque from N_2 in this frame is 0, but it is a non-inertial frame and students must therefore proceed with caution.

$$\tau = \tau_{mg} + \tau_{N_1} + \tau_{\substack{non-\\inertial}} = I_3\ddot{\theta} \qquad \text{(1 point)}$$

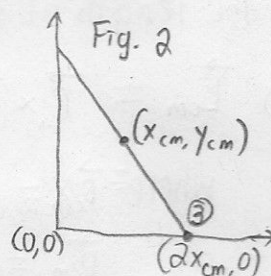Deduct $\frac{1}{2}$ point if $\tau_{non-in.}$ is not included.

$$\tau_{mg} = mg\ell\cos\theta \quad \text{out of page} \qquad \text{(1 point)}$$

$$\tau_{N_1} = 2N_2\ell\sin\theta \quad \text{into page} \qquad \text{(1 point)}$$

$$\tau_{non-in.} = \vec{r} \times \vec{F}_{Fictitious}$$

As seen in Fig. 2, the coordinates of point ③ are $(2x_{cm}, 0)$; therefore $\vec{F}_{fict} = -2\ddot{x}_{cm}m$

Fig. 2

So $\tau_{non-in.} = m(-x_{cm}, y_{cm}, 0) \times (-2\ddot{x}_{cm}, 0, 0)$

$$= 2m\ddot{x}_{cm}y_{cm} \quad \text{out of page} \qquad \text{(1 point)}$$

Using the force equation in the x direction, this yields:

$$\tau_{non-inert.} = 2N_2 y_{cm} = 2N_2\ell\sin\theta \quad \text{out of page}$$

$$I_3 = I_{cm} + I_{\substack{parallel\\axis}}$$

$$= \frac{m}{12}(2\ell)^2 + m\ell^2 = \frac{4}{3}m\ell^2$$

(1 point for I_{cm})
(1 point for parallel axis theorem)

Deduct $\frac{1}{2}$ point if the student uses $\frac{1}{12}m\ell^2$ instead.

Putting this all together, and noting that torque into the page corresponds to positive $\ddot{\theta}$, yields: (2 points)

$$\boxed{-mg\ell\cos\theta + 2N_2\ell\sin\theta - 2N_2\ell\sin\theta = -mg\ell\cos\theta = \frac{4}{3}m\ell^2\ddot{\theta}}$$

Deduct 1 point for using the incorrect signs.

Devote 2 points to rewriting x_{cm} and y_{cm} in terms of θ, either in this part or part d.

· Conservation of energy (this may be written in part d)

- Translational kinetic energy: $\frac{1}{2}m\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right)$ ($\frac{1}{2}$ point)
- Rotational kinetic energy : $\frac{1}{2}I_{cm}\omega^2 = \frac{1}{6}m\ell^2\dot{\theta}^2$ (1 point)
- Potential energy: placing the zero of the energy on the floor, the initial potential energy is $U = \frac{mgh}{2}$ while at height y_{cm}, $U = mgy_{cm}$ (1 point)

Therefore, $\frac{1}{2}m\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right) + \frac{1}{6}m\ell^2\dot{\theta}^2 + mgy_{cm} = \frac{mgh}{2}$

d. $\boxed{6.5 \text{ Total Points}}$

I will solve this part using the 3 methods given in part a), as well as an alternative 4th approach. Common to the approaches are the following observations:

- When the ladder leaves the wall, $\ddot{x}_{cm} = 0 = N$, (1 point)

- The energy equation can be rewritten all in terms of θ:

$$\frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{6}m\ell^2\dot{\theta}^2 + mg\ell\sin\theta = \frac{mgh}{2}$$

$$\text{So, } \dot{\theta}^2 = \frac{3g}{a\ell^2}\left(\frac{h}{2} - \ell\sin\theta\right) \qquad (1 \text{ point})$$

Plugging into $\ddot{x}_{cm} = \ell\left(-\sin\theta\,\ddot{\theta} - \cos\theta\,\dot{\theta}^2\right) = 0$ yields:

$$\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2 = 0 = \sin\theta\,\ddot{\theta} + \frac{3g}{a\ell}\left(\frac{h}{2} - \ell\sin\theta\right)\cos\theta \qquad (\text{Eq.1})$$

So it remains to solve for $\ddot{\theta}$. Devote 3 points to this:

Method #1: Note that $N_1 = 0$, so the torque equation is:

$$\tfrac{2}{3} m l^2 \ddot{\theta} = 2 N_2 l \cos\theta - m g l \cos\theta \qquad \circledast$$

$$N_2 = m \ddot{y}_{cm} + mg = m(l \cos\theta \ddot{\theta} - l \sin\theta \dot{\theta}^2 + g)$$

using the condition that $\ddot{x}_{cm} = 0$ implies:

$$\cos\theta \dot{\theta}^2 = -\sin\theta \ddot{\theta} \rightarrow \dot{\theta}^2 = -\tan\theta \ddot{\theta}$$

So $N_2 = m\left(\dfrac{l}{\cos\theta}(\cos^2\theta \ddot{\theta} + \sin^2\theta \ddot{\theta}) + g\right)$

$$= m\left(\dfrac{l}{\cos\theta} \ddot{\theta} + g\right)$$

Therefore, $\circledast$ becomes:

$$\tfrac{2}{3} m l^2 \ddot{\theta} = 2 m l^2 \ddot{\theta} + m g l \cos\theta$$

$$\rightarrow \ddot{\theta} = -\dfrac{3g}{4l} \cos\theta$$

Method #2: Note that $N_1 = 0$, so the torque equation is:

$$\tfrac{1}{3} m l^2 \ddot{\theta} = -N_2 l \cos\theta \quad \leftarrow \text{plug in form for } N_2 \text{ found above}$$

$$= -m g l \cos\theta - m l^2 \ddot{\theta}$$

$$\rightarrow \ddot{\theta} = -\dfrac{3g}{4l} \cos\theta$$

Method #3: The torque equation immediately yields:

$$\ddot{\theta} = -\dfrac{3g}{4l} \cos\theta$$

<u>Method #4</u>: A clever way to avoid the need for the torque equations entirely is to differentiate the energy equation:

$$\frac{d}{dt}\left(\frac{2}{3}ml^2\dot{\theta}^2 + mgl\sin\theta\right) = \frac{d}{dt}(mgh/2)$$

$$\frac{4}{3}ml^2\dot{\theta}\ddot{\theta} - mgl\cos\theta\dot{\theta} = 0$$

$$\rightarrow \ddot{\theta} = \frac{3g}{4l}\cos\theta$$

As seen above, all 4 methods lead to the same conclusion. This expression for $\ddot{\theta}$ can be plugged into Eq. 1 to yield: (1.5 points)

$$-\frac{3g}{4l}\sin\theta\cos\theta + \frac{3g}{2l^2}\left(\frac{h}{2} - l\sin\theta\right)\cos\theta = 0$$

$$-l\sin\theta + h - 2l\sin\theta = 0 \rightarrow h = 3l\sin\theta$$

the y coordinate of the top of the ladder is $2l\sin\theta$, so when the ladder leaves the wall this height is $\boxed{\frac{2}{3}h}$

2. The Hohmann Transfer $\boxed{20 \text{ points}}$

a. $\boxed{5 \text{ points total}}$

Total energy = kinetic energy + potential energy

$$= \tfrac{1}{2}mv^2 - \frac{GM_e m}{r} \qquad \text{(1 point)}$$

where M_e is the mass of the Earth.

We want to express this in terms of r alone.

For a circular orbit, the centripetal force obeys:

$$\frac{mv^2}{r} = F_{gravity} = \frac{GM_e m}{r^2} \qquad \text{(1 point)}$$

$$\rightarrow \tfrac{1}{2}mv^2 = \tfrac{1}{2}\frac{GM_e m}{r}$$

$$\rightarrow E_{tot} = -\frac{GM_e m}{2r} \qquad \text{(1 point)} \qquad [\text{Eq. 1}]$$

Applying to this particular situation:

- Initially, $r = 2R$, so $E_{initial} = -\dfrac{GM_e m}{4R}$

- Final orbit: $r = 4R$, so $E_{final} = -\dfrac{GM_e m}{8R}$

- Therefore, $\boxed{\Delta E = \dfrac{GM_e m}{8R}}$ (2 points for reaching this conclusion)

Note for grading:

To arrive at the functional form of Eq. 1, students may also invoke the Virial Theorem, which states that $2K = -U$ for a circular orbit. To receive full credit, they must state the theorem and explain why it applies to this situation.

b. [7 points total]

As noted in the hint, this problem is best approached using the conservation of angular momentum. I show two methods below.

Method #1

The general equation for the energy of a satellite is:

$$E_{tot} = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \qquad [\text{Eq. 2}]$$

Where I have used the satellite mass m, in place of the reduced mass $\frac{mM}{m+M}$, in the limit $m \ll M$.

At the minimum and maximum distances, there is no radial motion, so $E = \frac{L^2}{2mr^2} - \frac{GMm}{r}$ (1 point for this idea)

At r_{min}: $E = \frac{L^2}{2mr_{min}^2} - \frac{GMm}{r_{min}}$ (1 point)

At r_{max}: $E = \frac{L^2}{2mr_{max}^2} - \frac{GMm}{r_{max}}$ (1 point)

Conservation of energy implies:

$$\frac{L^2}{2mr_{min}^2} - \frac{GMm}{r_{min}} = \frac{L^2}{2mr_{max}^2} - \frac{GMm}{r_{max}} \qquad (2 \text{ points})$$

$$\Rightarrow \quad \frac{L^2}{2m}\left(\frac{1}{r_{min}^2} - \frac{1}{r_{max}^2}\right) = GMm\left(\frac{1}{r_{min}} - \frac{1}{r_{max}}\right)$$

$$\frac{L^2}{2m} = GMm\left(\frac{r_{max}-r_{min}}{r_{min}r_{max}}\right)\left(\frac{r_{min}^2 r_{max}^2}{r_{max}^2 - r_{min}^2}\right)$$

$$= GMm\left(\frac{r_{min}r_{max}}{r_{min}+r_{max}}\right)$$

Therefore, plugging into the expression for E at r_{min}:

$$E = \frac{GMm\ r_{max}}{r_{min}(r_{max}+r_{min})} - \frac{GMm}{r_{min}}$$

$$= -\frac{GMm}{r_{min}}\left(1 - \frac{r_{max}}{r_{max}+r_{min}}\right) = \frac{-GMm}{r_{min}+r_{max}} = \frac{-GMm}{2a}$$

Award **2 points** for arriving at this conclusion. Deduct 1 point if the final answer includes r_{min} or r_{max}, i.e., is not fully simplified.

Method #2

In this approach, I avoid the use of the formula in Eq. 2, which might not be familiar to students.

Instead, I will write the energy as in part a):

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

This holds for all points in the orbit.

As for conservation of momentum, note that $\vec{L} = \vec{r} \times \vec{p}$.

At minimum and maximum distances, $\vec{r} \perp \vec{p}$, so

$$|\vec{L}| = |\vec{r}||\vec{p}| = mvr \qquad \text{(1 point for this idea)}$$

At r_{min}: $\quad L = mv_1 r_{min}$ $\qquad (\frac{1}{2}$ point$)$

At r_{max}: $\quad L = mv_2 r_{max}$ $\qquad (\frac{1}{2}$ point$)$

Therefore, $\quad v_1 = \dfrac{v_2 r_{max}}{r_{min}}$

Plugging into the expressions for E yields:

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_{min}} = \frac{1}{2}m\frac{v_2^2 r_{max}^2}{r_{min}^2} - \frac{GMm}{r_{min}} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_{max}} \qquad \text{(1 point)}$$

With some rearranging, we find:

$$\frac{1}{2}mv_2^2 = GMm\left(\frac{1}{r_{min}}-\frac{1}{r_{max}}\right)\left(\frac{r_{min}^2}{r_{max}^2-r_{min}^2}\right)$$

$$= \frac{GMm}{r_{min}+r_{max}}\,\frac{r_{min}}{r_{max}} \qquad (2 \text{ points})$$

Therefore, plugging into the expression for E:

$$E = \frac{GMm}{r_{min}+r_{max}}\,\frac{r_{min}}{r_{max}} - \frac{GMm}{r_{max}}$$

$$= -\frac{GMm}{r_{max}}\left(1-\frac{r_{min}}{r_{min}+r_{max}}\right) = \frac{-GMm}{r_{min}+r_{max}} = \frac{-GMm}{2a} \qquad (2 \text{ points})$$

Deduct 1 point if the student fails to simplify, i.e. if the final result includes r_{min} or r_{max}.

C. ⬚ 6 points total

Remark: the problem statement informs students that they may define a dimensionally correct expression for the energy of a satellite in elliptical orbit in the case that they cannot solve part b. In that case, the general approach should still match that given here.

Let E_1 denote the energy of the circular orbit at radius $2R$; E_2 denote the energy of the semi-elliptical orbit of semi-major axis $a = \frac{2R+4R}{2} = 3R$; and E_3 denote the energy of the circular orbit of radius $4R$

- First speed boost: to go from energy E_1 to E_2

$$E_1 = -\frac{GMm}{4R} = \frac{1}{2}m\vec{v}_1^2 - \frac{GMm}{2R}$$

$$E_2 = -\frac{GMm}{6R} = \frac{1}{2}m|\vec{v_2}|^2 - \frac{GMm}{2R}$$

Where I have used the notation $|\vec{v_i}|^2$ to emphasize that these are magnitudes.

The change in speed is $\Delta|\vec{v}| = |\vec{v_2}| - |\vec{v_1}|$.
So, let's solve for $|\vec{v_2}|$ and $|\vec{v_1}|$ (2 points)

$$|\vec{v_1}| = \sqrt{\frac{GMm}{2R} - \frac{GMm}{4R}}\sqrt{\frac{2}{m}} = \sqrt{\frac{GM}{2R}} \quad \text{(1 point)}$$

$$|\vec{v_2}| = \sqrt{\frac{GMm}{2R} - \frac{GMm}{6R}}\sqrt{\frac{2}{M}} = \sqrt{\frac{2GM}{3R}} \quad \text{(1 point)}$$

Therefore, $\Delta|\vec{v}|_{1\to2} = \sqrt{\frac{GM}{2R}}\left(\frac{2}{\sqrt{3}} - 1\right)$ (1 point)

– Second speed boost: to go from energy E_2 to E_3

$$E_2 = \frac{-GMm}{6R} = \frac{1}{2}m|\vec{v_3}|^2 - \frac{GMm}{4R}$$

$$\Rightarrow |\vec{v_3}| = \sqrt{\frac{GMm}{R}\left(\frac{1}{4} - \frac{1}{6}\right)}\sqrt{\frac{2}{m}} = \sqrt{\frac{GM}{6R}} \quad \text{(1 point)}$$

[Note that we could also arrive at this using the relation $|\vec{v_2}| = 2|\vec{v_3}|$ from conservation of angular momentum.]

$$E_3 = \frac{-GMm}{8R} = \frac{1}{2}m|\vec{v_4}|^2 - \frac{GMm}{4R}$$

$$\Rightarrow |\vec{v_4}| = \sqrt{\frac{GMm}{R}\left(\frac{1}{4} - \frac{1}{8}\right)}\sqrt{\frac{2}{m}} = \sqrt{\frac{GM}{4R}} \quad \text{(1 point)}$$

Therefore, $\Delta|\vec{v}|_{3\to4} = \sqrt{\frac{GM}{4R}}\left(1 - \sqrt{\frac{2}{3}}\right)$ (1 point)

Note: a common mistake may be to find $\Delta\sqrt{v_f^2 - v_i^2}$ which does not equal $(v_f - v_i)$! Award a maximum of 2 points in this case.

d. $\boxed{\text{2 points total}}$

The key observation here is that $\Delta E_{tot} = \frac{8Mm}{8R}$, as found in part a. Indeed:

$$(E_3 - E_2) + (E_2 - E_1) = E_3 - E_1 = \frac{8Mm}{8R}$$

$\underset{\substack{\text{for second} \\ \text{boost}}}{\downarrow} \qquad \underset{\substack{\text{for first} \\ \text{boost}}}{\downarrow}$

What makes the Hohmann Transfer is that it minimizes $\Delta|\vec{v}|$ by making the velocity boost parallel to the initial velocity.

Problem 3 **(15 points total)**

a) **(5pt)**

Criteria for complete solutions:

- Described the relation as $PV \propto T$ or analogous (5pt)
- Suggested some other mathematical expression that works for the examples (5pt)

Cumulative partial credits for incomplete solutions:

- Showed equations/expressions that work for at least one case in the table (1pt)
- Showed equations/expressions that correlate 2 of the quantities correctly, but didn't summarize them into one expression (2pt)

b) **(10pt)**

Cumulative criteria for partial credit:

- Proposed a correct and feasible experiment to test the correlation between 2 quantities (3pt per experiment)
- Explicitly said what hypothesis was being tested/the correlation between which quantities (2pt per experiment)

Obs: the grader should be very lenient when judging experimental set up descriptions. Measuring instruments do not need to be mentioned, but from the description it should be clear what tool is maintaining P, V or T constant in the proposed experiment (e.g. rigid box, piston, atmosphere). Also, at least two experiments are necessary to test the correlations between the three variables.

Problem 4: Momentum in E-M Fields   Total points: 15

a. 2 Total points

$\vec{E}$ points in the $\hat{z}$ direction, $\vec{B}$ in the $\hat{x}$ direction
Momentum density $\mathcal{P} = \varepsilon_0 (\vec{E} \times \vec{B})$, which points in the $\hat{y}$
direction.

$\mathcal{P}$ is constant, so total momentum $\vec{p}$ is just $\mathcal{P} \times$ volume :

$$\boxed{\vec{P} = \varepsilon_0 |\vec{E}||\vec{B}| A d \,\hat{y}}$$

1 point for magnitude
1 point for direction
Deduct a full 1 point if the
student forgets to multiply by volume.

b. 5 Total Points

$$\vec{F} = \int I d\vec{\ell} \times \vec{B}$$

Current flows from the bottom plate to the top plate,
since $\vec{E}$ originally points in the $+\hat{z}$ direction.
So $I d\vec{\ell} \times \vec{B} = I |\vec{B}| d\ell \,\hat{y}$

This force is constant along the length of the wire, d
so $\vec{F} = I |\vec{B}| d \,\hat{y}$ (1 point)

The impulse delivered to the system is therefore:

$$\hat{y} |\vec{B}| d \int I \, dt = \hat{y} |\vec{B}| d \int -\frac{dQ}{dt} \, dt = -\hat{y} |\vec{B}| d \int_{Q_0}^{0} dQ$$
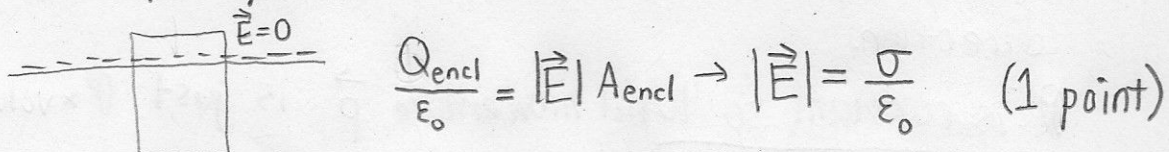
where Q is the charge on the bottom plate,
and Q_0 is the initial charge on the plate.

Therefore, $\vec{J} = -|\vec{B}| d Q_0 \,\hat{y}$ (1 point for magnitude)
                                               (1 point for direction)

To prove that this impulse equals the momentum originally contained in the field, it suffices to express Q_0 in terms of $|\vec{E}|$.

Students may state without proof that $|\vec{E}| = \dfrac{\sigma}{\varepsilon_0} = \dfrac{Q_0}{\varepsilon_0 A}$,
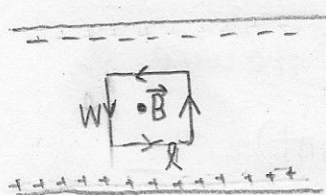
or they may use Gauss's Law to derive this:

$$\frac{Q_{encl}}{\varepsilon_0} = |\vec{E}| \, A_{encl} \rightarrow |\vec{E}| = \frac{\sigma}{\varepsilon_0} \qquad \text{(1 point)}$$

Therefore, $Q_0 = \varepsilon_0 A |\vec{E}|$, which implies :   (1 point for reaching this conclusion)

$$\vec{J} = \varepsilon_0 A d \, |\vec{B}| |\vec{E}| \, \hat{y} \qquad \text{which indeed agrees with part a).}$$

c. $\boxed{\text{8 Total Points}}$

Consider a rectangular loop of length ℓ, width w perpendicular to the plates, as shown below:

$\vec{B}$ points out of the page, so $\vec{B} \cdot d\vec{A}$

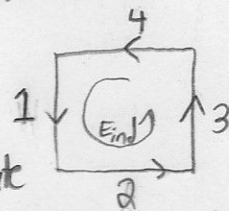$= |\vec{B}| \, dA$ (where the direction of $d\vec{A}$ is determined by the orientation of the loop, as shown at left).

Therefore, $-\displaystyle\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} = -\frac{d}{dt}\left(|\vec{B}| \ell w\right) = \ell w \left(\frac{-d|\vec{B}|}{dt}\right)$

Students may also write this as $-\dfrac{d\Phi_B}{dt}$, where Φ_B

is the flux through the loop.   (2.5 points)

Now take $w \to d$ so the horizontal legs of the loop are on the plates.

According to Faraday's law of induction, this equals $\oint \vec{E}_{ind} \, d\vec{\ell}$.

$\vec{E}_{ind}$ along leg 1 and leg 3 cannot contribute to this integral since this would require different magnitudes or directions of force on different parts of the same plate.

The only contributions to the integral are from legs 2 and 4, where $\vec{E}$ points in opposite directions with equal magnitude: (in the $-\hat{y}$ direction of top, $+\hat{y}$ direction on bottom)

$$|\vec{E}_{ind}|(2\ell) = -\ell d \frac{d|\vec{B}|}{dt}$$

or $|\vec{E}_{ind}| = -\frac{d}{2}\frac{d|\vec{B}|}{dt}$ (2.5 points)

So the force on the top plate is $(-Q_0)\left(+\frac{d}{2}\frac{d|\vec{B}|}{dt}\right)\hat{y}$ while the force on the bottom plate is $(+Q_0)\left(-\frac{d}{2}\frac{d|\vec{B}|}{dt}\right)\hat{y}$ (1 point)

so the total force on the plates is: (2 points)

$$-Q_0 d \frac{d|\vec{B}|}{dt}\hat{y} = -\varepsilon_0 A d |\vec{E}|\frac{d|\vec{B}|}{dt} \quad \text{using the result from b).}$$

Therefore, $\vec{J} = -\hat{y}\varepsilon_0 A d |\vec{E}| \int \frac{d|\vec{B}|}{dt} dt$

$= \varepsilon_0 A d |\vec{E}| |\vec{B}|\hat{y}$ (1 point)

which agrees with a) and b).